PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

SOLUTIONS FOR ASSIGNMENT #9

Reading Assignments:
Sections 4.8 and 5.1 of Peskin and Schroeder.

Problem 1
Do Problem 4.3 in Peskin and Schroder.
Solution:
(a) The potential includes the following 4-point interaction vertices: $(\lambda/4)(\Phi^i)^4 + (\lambda/2)(\Phi^i)^2(\Phi^j)^2$. If all four external particles are identical, $i = j = k = l$, only the contraction with the first type of vertex is non-vanishing and the symmetry factor is $4! \times (-i\lambda/4) = -6i\lambda$, because there are 4! ways to contract the four identical bosons. On the other hand, when the external particles are a pair of identical bosons, the symmetry factor is $2^2 \times (-i\lambda/2) = -2i\lambda$ since there are four different ways to contract with the second type of vertex.

The differential cross-section in the CM frame is given in Eq. (4.85) in Peskin and Schroeder

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{A}|^2}{64\pi^2 E_{\text{cm}}^2}.$$

Using the Feynman rules it is easy to obtain the amplitudes for the three different scattering processes: $i\mathcal{A}(\phi^1\phi^2 \to \phi^1\phi^2) = -2i\lambda$, $i\mathcal{A}(\phi^1\phi^1 \to \phi^2\phi^2) = -2i\lambda$, and $i\mathcal{A}(\phi^1\phi^1 \to \phi^1\phi^1) = -6i\lambda$.
(b) Since the VEV minimizes the potential, the vanishing of first derivatives requires $v^2 = \mu^2/\lambda$. Then we have

$$\begin{aligned} V &= -\frac{1}{2}\mu^2\left[(\pi^i)^2 + (v+\sigma)^2\right] + \frac{\lambda}{4}\left[(\pi^i)^2 + (v+\sigma)^2\right]^2 \\ &= -\frac{\mu^2}{4\lambda} + \mu^2\sigma^2 + \mu\sqrt{\lambda}\sigma^3 + \frac{\lambda}{4}\sigma^4 + \frac{\lambda}{4}\left((\pi^i)^2\right)^2 + \sqrt{\lambda}\mu\sigma(\pi^i)^2 + \frac{\lambda}{2}\sigma^2(\pi^i)^2. \end{aligned}$$

Therefore σ is massive with a mass $\sqrt{2}\mu$ and the π^i is massless.

Now we come to the Feynman rules. The 4-pt vertex with four pions is the same as in (a), 4-pt vertex with two σ-fields and two pion fields $\pi^i\pi^j$ is $-2i\lambda\delta^{ij}$, and 4-pt vertex with all four σ-field is $-6i\lambda$. For 3-pt vertices, all three with σ-field is $-6i\lambda v = -6i\sqrt{\lambda}\mu$, and with one σ-field and two pion-fields $\pi^i\pi^j$ is $-2i\lambda v = -2i\mu\sqrt{\lambda}\delta^{ij}$.
(c) The amplitude for $\pi^i(k_1)\pi^j(k_2) \to \pi^k(p_1)\pi^l(p_2)$ is

$$\begin{aligned} i\mathcal{A} &= \frac{-4i\lambda^2v^2}{(k_1+k_2)^2 - 2\mu^2}\delta^{ij}\delta^{kl} + \frac{-4i\lambda^2v^2}{(k_1-p_1)^2 - 2\mu^2}\delta^{ik}\delta^{jl} + \frac{-4i\lambda^2v^2}{(k_1-p_2)^2 - 2\mu^2}\delta^{il}\delta^{kj} \\ &\quad -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{kj}). \end{aligned}$$

At threshold $k_1 = k_2 = p_1 = p_2 = 0$ because the pion is massless. Then one sees $\mathcal{A} = 0$. For the special case $N = 2$, there is only one species of pions, the amplitude is

$$i\mathcal{A} = \frac{-4i\lambda\mu^2}{(k_1+k_2)^2 - 2\mu^2} + \frac{-4i\lambda\mu^2}{(k_1-p_1)^2 - 2\mu^2} + \frac{-4i\lambda\mu^2}{(k_1-p_2)^2 - 2\mu^2} - 2i\lambda.$$

When the momentum is small comparing to the mass μ, we can expand the propagator in the large μ limit to $\mathcal{O}(p^2)$:

$$i\mathcal{A} = \frac{i\lambda}{\mu^2}\left[(k_1+k_2)^2 + (k_1-p_1)^2 + (k_1-p_2)^2\right] + \mathcal{O}(p^4).$$

Then using the momentum conservation $k_1 + k_2 = p_1 + p_2$ as well as the massless condition $k_1^2 = k_2^2 = p_1^2 = p_2^2 = 0$, we see the $\mathcal{A} = 0$ at order p^2.
(d) The potential and the minimization condition is now

$$\begin{aligned} V &= -\frac{1}{2}\mu^2(\Phi^i)^2 + \frac{\lambda}{4}((\Phi^i)^2)^2 - a\Phi^N \\ \frac{\partial V}{\partial \Phi^j} &= -\mu^2\Phi^j + \lambda\Phi^j(\Phi^i)^2 - a\delta^{Nj} \end{aligned}$$

The solution is $\langle\Phi^i\rangle = \delta^{Ni}v$, where v satisfies $-\mu^2 v + \lambda v^3 - a = 0$. Treating a as a small parameter we have $v = \mu/\sqrt{\lambda} + a/(2\mu^2)$. The spectrum of the theory is obtained by computing the second derivative of the potential: $m_\sigma^2 = \partial^2 V(0)/\partial\sigma^2 = 2\mu^2 + 3a\sqrt{\lambda}/\mu$ and $m_\pi^2 = \partial^2 V(0)/\partial\pi^2 = a\sqrt{\lambda}/\mu$. So indeed the pion gets a mass proportional to a.

The Feynman rule also gets corrections due to a, which again can be obtained from looking at the quartic term in the potential

$$\frac{\lambda}{4}((\pi^i)^2 + (v+\sigma)^2)^2$$

The 4-pt vertex comes from computing $\partial^4 V/\partial\phi^4$ and is not modified. The 3-pt vertex, however, does get modified. The $\pi^i - \pi^j - \sigma$ vertex comes from $\partial^3 V/\partial\pi^i\partial\pi^j\partial\sigma$ and the $\sigma - \sigma - \sigma$ vertex from $\partial^3 V/\partial\sigma^3$. But then we see the modification is simply replacing the old v in (b) by the new $v = \mu/\sqrt{\lambda} + a/(2\mu^2)$. This observation makes it easy to compute the scattering amplitude by replacing v in the amplitude in (c) with the new v as well as the mass of the σ with the new mass:

$$\begin{aligned} i\mathcal{A} &= \frac{-4i\lambda^2 v^2}{(k_1+k_2)^2 - 2\mu^2 - 3a\sqrt{\lambda}/\mu}\delta^{ij}\delta^{kl} + \frac{-4i\lambda^2 v^2}{(k_1-p_1)^2 - 2\mu^2 - 3a\sqrt{\lambda}/\mu}\delta^{ik}\delta^{jl} \\ &\quad + \frac{-4i\lambda^2 v^2}{(k_1-p_2)^2 - 2\mu^2 - 3a\sqrt{\lambda}/\mu}\delta^{il}\delta^{kj} - 2i\lambda(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{kj}). \end{aligned}$$

Now at threshold $k_1^2 = k_2^2 = p_1^2 = p_2^2 = m_\pi^2 = a\sqrt{\lambda}/\mu$, $(k_1+k_2)^2 = 4m_\pi^2$, $(k_1-p_1)^2 = (k_1-p_2)^2 = 0$. So now we have

$$\begin{aligned} i\mathcal{A} &= 2i\lambda\left[\left(1 + \frac{3}{2}\frac{a\sqrt{\lambda}}{\mu^3}\right)\delta^{ij}\delta^{kl} + \left(1 - \frac{1}{2}\frac{a\sqrt{\lambda}}{\mu^3}\right)(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{kj})\right] \\ &\quad -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{kj}) \\ &= 2i\lambda\left[\frac{3}{2}\frac{a\sqrt{\lambda}}{\mu^3}\delta^{ij}\delta^{kl} - \frac{1}{2}\frac{a\sqrt{\lambda}}{\mu^3}(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{kj})\right]. \end{aligned}$$

Indeed, it is nonvanishing and proportional to a.

Problem 2
Consider a complex scalar field ϕ coupling to the muon (μ^-) field μ and the neutrino (ν) field ν_L through the following interaction

$$\mathcal{L}_I = C_F(\partial_\mu\phi)(\bar{\mu}\gamma^\mu\nu_L) + (\text{h.c.}),$$

where we have assumed the neutrino is massless and exists only in the left-handed component $\nu_L = (1/2)(1-\gamma_5)\nu$. The complex scalar ϕ has a mass m_ϕ and the muon has a mass m_μ.
(a) Compute the differential cross-secion $d\sigma/d\Omega$ for the scattering process $\bar{\nu}+\nu \to \mu^+ + \mu^-$ to leading order in C_F in the centre-of-mass frame.
(b) Compute the total decay width Γ of the complex scalar to leading order in C_F.
(c) We can think of the complex scalar ϕ as the charged pion π^-. In the standard model the coefficient of the interaction is

$$C_F = 2\cos\theta_c\, G_F\, f_\pi$$

where $\cos\theta_c = 0.974$ is the cosine of the Cabbibo angle, $G_F = 1.166\times 10^{-5}$ GeV^{-2} is the Fermi constant, and f_π is the pion decay constant. In addition, the pion mass is $m_\pi = 139.6$ MeV and the muon mass $m_\mu = 105.7$ MeV. The measured value of the charged pion lifetime is $2.603\times 10^{-8}s$. Determine the value of f_π in MeV. By what percentage is your result too big or too small?
Solution:
(a) Using the momentum and spin assignment $\nu(k_1,r)+\bar{\nu}(k_2,s)\to\mu^-(p_1,r')+\mu^+(p_2,s')$, the amplitude is

$$\begin{aligned}
i\mathcal{A} &= \frac{|C_F|^2}{4}\bar{u}^{r'}(p_1)(-\not{k}_1+\not{p}_1)(1-\gamma_5)u^r(k_1)\frac{i}{(k_1-p_1)^2-m_\phi^2+i\epsilon}\bar{v}^{s'}(p_2)(\not{p}_2-\not{k}_2)(1-\gamma_5)v^s(k_2)\\
&= \frac{|C_F|^2}{4}(-m_\mu^2)\bar{u}^{r'}(p_1)(1-\gamma_5)u^r(k_1)\frac{i}{(k_1-p_1)^2-m_\phi^2+i\epsilon}\bar{v}^{s'}(p_2)(1-\gamma_5)v^s(k_2)
\end{aligned}$$

where we have used Dirac equations to simplify the numerator in getting to the second line. After summing over final spins and averaging over initial spins, the square of the amplitude is then

$$\begin{aligned}
&\frac{1}{4}\sum_{r,s}\sum_{r',s'}|\mathcal{A}|^2 =\\
&\quad\frac{C_F^4}{64}\frac{m_\mu^4}{[(k_1-p_1)^2-m_\phi^2]^2}\mathrm{Tr}\left[(\not{p}_1+m_\mu)(1-\gamma^5)\not{k}_1(1+\gamma^5)\right]\mathrm{Tr}\left[(\not{p}_2-m_\mu)(1-\gamma^5)\not{k}_2(1+\gamma^5)\right]\\
&= \frac{C_F^4 m_\mu^4}{[(k_1-p_1)^2-m_\phi^2]^2}(p_1\cdot k_1)(p_2\cdot k_2).
\end{aligned}$$

In the CM frame we define the Lorentz boost factors $\beta = v, \gamma = 1/\sqrt{1-\beta^2}$, then the kinematics are given as follows: $p_1 = (m_\mu\gamma, m_\mu\gamma\beta\hat{p})$, $p_2 = (m_\mu\gamma, -m_\mu\gamma\beta\hat{p})$, $k_1 = (m_\mu\gamma, m_\mu\gamma\hat{z})$, and $k_2 = (m_\mu\gamma, -m_\mu\gamma\hat{z})$. Also define $\cos\theta = \hat{z}\cdot\hat{p}$ so that $k_1\cdot p_1 = k_2\cdot p_2 = m_\mu^2\gamma^2(1-\beta\cos\theta)$. Then the amplitude-squared in the CM frame is

$$\frac{1}{4}\sum_{r,s}\sum_{r',s'}|\mathcal{A}|^2 = C_F^4 m_\mu^4\frac{\gamma^4(1-\beta\cos\theta)^2}{(1-2\gamma^2(1-\beta\cos\theta)-m_\phi^2/m_\mu^2)^2}$$

As a simple check, the amplitude should be dimensionless, which indeed the case since $[C_F] = M^{-1}$. The differential cross-section is

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{2E_{cm}^2}\frac{m_\mu\gamma\beta}{(2\pi)^2 4E_{cm}}\frac{1}{4}\sum_{r,s}\sum_{r',s'}|\mathcal{A}|^2 \\ &= \frac{C_F^4 m_\mu^2}{256\pi^2}\frac{\gamma^2(1-\beta\cos\theta)^2}{(1-2\gamma^2(1-\beta\cos\theta)-m_\phi^2/m_\mu^2)^2}\end{aligned}$$

where $E_{cm} = 2m_\mu\gamma$. Again we see the differential cross-section has the dimension of L^2.
(b) For $\phi(k) \to \nu(p_1, r) + \mu^-(p_2, s)$, the amplitude is

$$\begin{aligned}i\mathcal{A} &= C_F\bar{u}^s(p_2)(\not{p}_1+\not{p}_2)\frac{1-\gamma_5}{2}u^s(p_1) \\ &= \frac{C_F m_\mu}{2}\bar{u}^s(p_2)(1-\gamma_5)u^s(p_1)\end{aligned}$$

where I have again used the Dirac equation to simplify the amplitude. The spin-summed amplitude-squared is

$$\sum_{r,s}|\mathcal{A}|^2 = \frac{C_F^2 m_\mu^2}{2}\mathrm{Tr}[(\not{p}_2+m_\mu)\not{p}_1(1+\gamma^5)] = 2C_F^2 m_\mu^2(p_1\cdot p_2).$$

In the CM frame $p_1 = (p, p\hat{p})$ and $p_2 = (\sqrt{p^2+m_\mu^2}, -p\hat{p})$. Energy conservation determines $p = (m_\phi^2 - m_\mu^2)/(2m_\phi)$, $\sqrt{p^2 - m_\mu^2} = (m_\phi^2 + m_\mu^2)/(2m_\phi)$, and $p_1 \cdot p_2 = (m_\phi^2 - m_\mu^2)/2$. We see that the ampliude-squared is isotropic in the CM frame and has no angular dependence. Then

$$\begin{aligned}\int d\Gamma &= \frac{1}{2m_\phi}\int d\Pi_2\sum_{r,s}|\mathcal{A}|^2 \\ &= \frac{p}{8\pi E_{cm} m_\phi}\sum_{r,s}|\mathcal{A}|^2 \\ &= \frac{C_F^2}{16\pi}\frac{m_\mu^2}{m_\phi^3}(m_\phi^2 - m_\mu^2)^2\end{aligned}$$

(c) $\tau = 1/\Gamma = 2.603\times10^{-8}\ s = 3.955\times10^{14}$ MeV. From (b) we get $f_\pi = 93.14$ MeV. Comparing with the measured $f_\pi = 92.42$ MeV, the result is too large by 0.7% due to the electromagnetic loop corrections.